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DEFORMATION RETRACTS AND THE HOCHSCHILD HOMOLOGY OF POLYNOMIAL RINGS

BY

AYELET LINDENSTRAUSS*

Department of Mathematics
Technion — Israel Institute of Technology, Haifa, Israel
e-mail: ayeletl@techunix.technion.ac.il

ABSTRACT

We assume given a ring A with unit, and a subcomplex of the reduced bar complex of A. We assume that this subcomplex is a deformation retract of the whole complex and thus has homology equal to the Hochschild homology of A, but it will typically be smaller and easier to calculate with. We use these to construct (accordingly small) deformation retracts for the reduced bar complexes of $A[t]$ and $A[t, t^{-1}]$. When A is a Banach algebra, we also do this construction for $C^\infty(S^1; A)$.

Introduction

In computations of Hochschild homology and its variants, such as cyclic homology, neither the standard Hochschild complex

$$\cdots \xrightarrow{d_3} A \otimes A \otimes A \xrightarrow{d_2} A \otimes A \xrightarrow{d_1} A \longrightarrow 0$$

nor the reduced Hochschild (bar) complex

$$(0.1) \qquad \cdots \xrightarrow{d_3} A \otimes \bar{A} \otimes \bar{A} \xrightarrow{d_2} A \otimes \bar{A} \xrightarrow{d_1} A \longrightarrow 0$$

is very often used. Typically, a much smaller direct summand of the complex, which is quasi-isomorphic to the whole, is used instead. For example, if A is a

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smooth n-dimensional commutative algebra over a commutative $\mathbb{Q}$-algebra k, the complex with zero differentials

$$0 \longrightarrow \Omega^n_{A/k} \xrightarrow{0} \Omega^{n-1}_{A/k} \xrightarrow{0} \cdots \xrightarrow{0} \Omega^1_{A/k} \xrightarrow{0} A \longrightarrow 0$$

can be realized as a subcomplex of (0.1) via $i\colon \Omega^r_{A/k} \longrightarrow A \otimes \bar{A}^{\otimes(r)}$ sending

$$f_0 \,\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^{\mathrm{sgn}(\sigma)} f_0 \otimes f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(r)}$$

and as a quotient of (0.1) via $\pi\colon A \otimes \bar{A}^{\otimes(r)} \longrightarrow \Omega^r_{A/k}$ sending

$$f_0 \otimes f_1 \otimes \cdots \otimes f_k \mapsto f_0 \,\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_r.$$

Clearly $\pi \circ i = \mathrm{id}_{\Omega^r_{A/k}}$, and by the Hochschild–Kostant–Rosenberg theorem, i and π are actually quasi-isomorphisms. Following Kassel [1], we define a deformation retract (of the reduced Hochschild complex) to be a complex $(X_., d_X)$ together with chain maps $i_.\colon X_. \longrightarrow A \otimes \bar{A}^{\otimes(\cdot)}$ and $\pi_.\colon A \otimes \bar{A}^{\otimes(\cdot)} \longrightarrow X_.$ and a chain homotopy $K_.\colon A \otimes \bar{A}^{\otimes(\cdot)} \longrightarrow A \otimes \bar{A}^{\otimes(\cdot+1)}$ such that

$$\pi_r \circ i_r = \mathrm{id}_{X_r}, \qquad d_{r+1} \circ K_r + K_{r-1} \circ d_r = \mathrm{id}_{A \otimes \bar{A}^{\otimes(r)}} - i_r \circ \pi_r.$$

Hitherto, there have been *ad hoc* computations of deformation retracts, but no attempt to systematize the construction. This paper is a step in the latter direction. Given a k-algebra A with unit and a deformation retract for A, we construct such retracts for $A[t]$ and $A[t, t^{-1}]$. The latter construction can be adapted to give, for each Banach algebra A and deformation retract of the *continuous* reduced Hochschild complex for A, a deformation retract for the ring $C^\infty[S^1; A]$ of smooth A-valued functions on the circle. It seems likely that similar methods will work for the non-commutative polynomial rings

$$A\{t\}/(at - t\phi(a)),$$

where ϕ is an automorphism of A. A more difficult question, which we do not discuss, is the construction of deformation retracts for $A/a \cdot A$, where a is an element of the center of A. It would be nice to have a general construction that specializes to Wolffhardt's complex for coordinate rings of local complete intersection.

Finally, it should be noted that the results of this paper do not advance the state of the art in computing Hochschild homology since the Künneth formula already expresses $\mathrm{HH}_*(A[t])$ and $\mathrm{HH}_*(A[t,t^{-1}])$ in terms of $\mathrm{HH}_*(A)$. The results which follow do, however, provide information about explicit homotopies that the Künneth formula does not. Already in the case of $k[x,y]$, a homotopy contracting the reduced Hochschild bar complex to the Hochschild-Kostant-Rosenberg complex is quite difficult to find. The methods of this paper give a general inductive construction.

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1. Going from a ring A to $A[t]$ and to $A[t,t^{-1}]$

In this section, we will assume given a commutative ground ring k with a unit,a k-algebra A, also with unit (so that if we multiply that unit by any element of k we get an element in the center of A), and a complex of k-modules $(X_., d_X)$ whose homology is $\mathrm{HH}_*(A)$. Typically such a complex arises by tensoring a projective resolution of A, as an $A \otimes A^{op}$-module, over $A \otimes A^{op}$ with A, but this will not be used in the calculations. What we do want is that the complex $X.$ should be a direct summand in the usual reduced bar complex calculating $\mathrm{HH}_*(A)$,

$$\cdots \xrightarrow{d_3} A \otimes \bar{A}^{\otimes(2)} \xrightarrow{d_2} A \otimes \bar{A} \xrightarrow{d_1} A \longrightarrow 0 \tag{1.1}$$

where $\bar{A} = A/(k \cdot 1)$, and

$$d_r \colon A \otimes \bar{A}^{\otimes(r)} \longrightarrow A \otimes \bar{A}^{\otimes(r-1)}$$

is given by

$$d_r = \sum_{i=0}^{r} (-1)^i d_{(i)},$$

$$d_{(i)}(a_0 \otimes a_1 \otimes \cdots \otimes a_r) = a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_r, \quad 0 \le i \le r-1,$$

$$d_{(r)}(a_0 \otimes a_1 \otimes \cdots \otimes a_r) = a_r a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}.$$

In some of the calculations, it will also be useful to define the operator $b' \colon A \otimes \bar{A}^{\otimes(s)} \longrightarrow A \otimes \bar{A}^{\otimes(s-1)}$ (or sometimes $b' \colon \bar{A}^{\otimes(s+1)} \longrightarrow \bar{A}^{\otimes(s)}$) given by

$$b' = \sum_{i=0}^{s-1} (-1)^i d_{(i)}. \tag{1.2}$$

Since the reduced bar complex is itself a direct summand in the original (non-reduced) bar complex, by having our complex $X_.$ be a direct summand in the reduced bar complex we automatically get it as a direct summand in the non-reduced complex; the reduced complex is, however, easier to work with.

The fact that $X_.$ is a direct summand in the reduced bar complex means that there are chain maps $i_.^{\mathrm{old}}$, $\pi_.^{\mathrm{old}}$,

$$i_r^{\mathrm{old}}\colon X_r \longrightarrow A \otimes \bar{A}^{\otimes(r)}, \tag{1.3}$$

$$\pi_r^{\mathrm{old}}\colon A \otimes \bar{A}^{\otimes(r)} \longrightarrow X_r, \tag{1.4}$$

such that

$$\pi_r^{\mathrm{old}} \circ i_r^{\mathrm{old}} = \mathrm{id}_{X_r},$$

$$i_r^{\mathrm{old}} \circ \pi_r^{\mathrm{old}} \simeq \mathrm{id}_{A\otimes\bar{A}^{\otimes(r)}}$$

for every r. Moreover, let us assume given a homotopy $K_.^{\mathrm{old}}$ between the two last expressions, $K_r^{\mathrm{old}}\colon A \otimes \bar{A}^{\otimes(r)} \longrightarrow A \otimes \bar{A}^{\otimes(r+1)}$ such that

$$d_{r+1} \circ K_r^{\mathrm{old}} + K_{r-1}^{\mathrm{old}} \circ d_r = \mathrm{id}_{A\otimes\bar{A}^{\otimes(r)}} - i_r^{\mathrm{old}} \circ \pi_r^{\mathrm{old}}. \tag{1.5}$$

We will use this data to obtain similar structures on the rings $A[t]$ and $A[t,t^{-1}]$ —in particular, we will exhibit direct summands in the reduced bar complexes which are quasi isomorphic to the whole respective complexes, but easier to calculate. The construction will be carried out simulatneously for $A[t]$ and $A[t,t^{-1}]$. Let $B = k[t]$, or respectively $B = k[t,t^{-1}]$.

In either case, the following complex is a free $B \otimes B$ resolution of B:

$$0 \longrightarrow B \otimes B \xrightarrow{\cdot(1\otimes t - t\otimes 1)} B \otimes B \xrightarrow{\text{mult}} B \longrightarrow 0. \tag{1.6}$$

After tensoring over $B \otimes B$ with B, we obtain the complex

$$0 \longrightarrow B \xrightarrow{0} B \longrightarrow 0.$$

We could tensor the small resolution of A which gave us $X_.$ with the small resolution of B in (1.6) to obtain a resolution of $A \otimes B$. Tensoring the resulting complex with $A \otimes B$, we obtain the following complex whose homology is $\mathrm{HH}_*(A \otimes B)$:

$$\tag{1.7} \begin{aligned} \cdots \xrightarrow{d} (X_3 \otimes B) \oplus (X_2 \otimes B)\,\mathrm{dt} \xrightarrow{d} (X_2 \otimes B) \oplus (X_1 \otimes B)\,\mathrm{dt} \\ \xrightarrow{d} (X_1 \otimes B) \oplus (X_0 \otimes B)\,\mathrm{dt} \xrightarrow{d} X_0 \otimes B \longrightarrow 0 \end{aligned}$$

where for all $\alpha \in X_.$ and $i \geq 0$,

$$\tag{1.8} d(\alpha t^i) = d_X(\alpha) t^i, \quad d(\alpha t^i\,\mathrm{dt}) = d_X(\alpha) t^i\,\mathrm{dt}\,.$$

dt in this context can be looked upon as a place marker of degree 1. Note that the homology of the complex (1.7) gives the Hochschild homology of $A \otimes B$ by construction in the case where $X_.$ comes from an $(A \otimes B) \otimes (A^{op} \otimes B)$-resolution of $A \otimes B$, but also works for any complex $X_.$ which is a direct summand in the reduced bar complex because of the calculations which follow.

For the new complex (1.7), we define maps

$$i_r^{\mathrm{new}}\colon (X_r \otimes B) \oplus (X_{r-1} \otimes B)\,\mathrm{dt} \longrightarrow A \otimes B \otimes \overline{(A \otimes B)}^{\,r}$$

given by

$$\tag{1.9} \begin{aligned} i_r^{\mathrm{new}}(\alpha_r t^i) &= t^i i_r^{\mathrm{old}}(\alpha_r) & \forall \alpha_r \in X_r, \quad i \geq 0, \\ i_r^{\mathrm{new}}(\alpha_{r-1} t^i\,\mathrm{dt}) &= t^i i_r^{\mathrm{old}}(\alpha_r) * (1 \otimes t) & \forall \alpha_{r-1} \in X_{r-1}, \quad i \geq 0, \end{aligned}$$

where $*$ is used to denote the shuffle product, and

$$\pi_r^{\mathrm{new}}\colon A \otimes B \otimes \overline{(A \otimes B)}^{\,r} \longrightarrow (X_r \otimes B) \oplus (X_{r-1} \otimes B)\,\mathrm{dt}$$

given by

(1.10)
$$\begin{aligned} &\pi_r^{\mathrm{new}}(a_0 t^{n_0} \otimes a_1 t^{n_1} \otimes \cdots \otimes a_r t^{n_r}) \\ &\qquad = t^{n_0+n_1+\cdots+n_r} \pi_r^{\mathrm{old}}(a_0 \otimes a_1 \otimes \cdots \otimes a_r) \\ &\qquad\qquad + n_r a_r t^{n_0+n_1+\cdots+n_r-1} \pi_{r-1}^{\mathrm{old}}(a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1})\,\mathrm{dt}\,. \end{aligned}$$

Note that in defining i_r^{new} we shuffled with $1 \otimes t$, and in defining π_r^{new} we had a term $n_r a_r t^{n_0+n_1+\cdots+n_r-1}$ in which we reduced the exponent by one. In the case of $B = k[t, t^{-1}]$, we could, of course, have shuffled with $1 \otimes t^{-1}$ and increased the exponent by one instead, but in making the choice the symmetry between t and t^{-1} is broken. The choice has been made in this way so that it would also make sense for $B = k[t]$, and so that our formulæ for $B = k[t, t^{-1}]$ will restrict to formulæ for $B = k[t]$ when we view $k[t]$ as embedded in $k[t, t^{-1}]$ in the standard way.

Direct calculation shows that $i_{\cdot}^{\text{new}}$, $\pi_{\cdot}^{\text{new}}$ are chain maps. Clearly,

$$\pi_r^{\text{new}} \circ i_r^{\text{new}}(\alpha_r t^i) = \pi_r^{\text{new}}(t^i i_r^{\text{old}}(\alpha_r)) = t^i \pi_r^{\text{old}} \circ i_r^{\text{old}}(\alpha_r) = \alpha_r t^i \tag{1.11}$$

and

$$\begin{aligned} \pi_r^{\text{new}} \circ i_r^{\text{new}}(\alpha_r t^i \,\mathrm{dt}) &= \pi_r^{\text{new}}(t^i i_r^{\text{old}}(\alpha_r) * (1 \otimes t)) \\ &= t^{i+1} \pi_r^{\text{old}}(i_r^{\text{old}}(\alpha_r) * (1 \otimes 1)) + t^i \pi_r^{\text{old}} \circ i_r^{\text{old}}(\alpha_r) \,\mathrm{dt} \\ &= \alpha_r t^i \,\mathrm{dt} \end{aligned} \tag{1.12}$$

so

$$\pi_r^{\text{new}} \circ i_r^{\text{new}} = \mathrm{id}_{(X_r \otimes B) \oplus (X_{r-1} \otimes B)\,\mathrm{dt}} \,. \tag{1.13}$$

To demonstrate that $i_r^{\text{new}} \circ \pi_r^{\text{new}} \simeq \mathrm{id}$, we will define an explicit homotopy $K_{\cdot}^{\text{new}}$, on the basis of the homotopy $K_{\cdot}^{\text{old}}$ which we had for the ring A, by

$$\begin{aligned} &K_r^{\text{new}}(a_0 t^{n_0} \otimes a_1 t^{n_1} \otimes \cdots \otimes a_r t^{n_r}) \\ &= \sum_{j=1}^{r} (-1)^j \begin{Bmatrix} \sum_{i=0}^{n_j - 1} & \text{if } n_j \geq 0 \\ \sum_{i=n_j}^{-1} & \text{if } n_j \leq 0 \end{Bmatrix} t^{n_0 + n_1 + \cdots + n_{j-1} + i} \\ &\qquad\qquad [(a_0 \otimes a_1 \otimes \cdots \otimes a_{j-1}) * (1 \otimes t)] \\ &\qquad\qquad \otimes a_j t^{n_j - 1 - i} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r} \\ &+ n_r a_r t^{n_0 + n_1 + \cdots + n_r - 1} K_{r-1}^{\text{old}}(a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}) * (1 \otimes t) \\ &+ t^{n_0 + n_1 + \cdots + n_r} K_r^{\text{old}}(a_0 \otimes a_1 \otimes \cdots \otimes a_r). \end{aligned} \tag{1.14}$$

CLAIM: $d_{r+1} \circ K_r^{\text{new}} + K_{r-1}^{\text{new}} \circ d_r = \mathrm{id}_{A \otimes \bar{A}^{\otimes(r)}} - i_r^{\text{new}} \circ \pi_r^{\text{new}}$.

Proof: Set

$$F = d_{r+1} \circ K_r^{\text{new}} + K_{r-1}^{\text{new}} \circ d_r - \mathrm{id} + i_r^{\text{new}} \circ \pi_r^{\text{new}}.$$

We will show that $F(a_0 t^{n_0} \otimes a_1 t^{n_1} \otimes \cdots \otimes a_r t^{n_r}) \equiv 0$ for all a_i and all the values of exponents n_i which appear in B. The proof will proceed by induction on r; for each r the proof will be by induction on $|n_r|, |n_{r-1}|, \ldots, |n_1|, |n_0|$, in that order.

THE CASE OF $r = 0$: In this case we must check

$$d_1 \circ K_0^{\text{new}}(a_0 t^{n_0}) \stackrel{?}{=} a_0 t^{n_0} - t^{n_0} \pi_0^{\text{old}} \circ i_0^{\text{old}}(a_0)$$

which holds since $d_1 \circ K_0^{\text{old}} = \text{id} - \pi_0^{\text{old}} \circ i_0^{\text{old}}$.

THE INDUCTIVE STEP, $r > 0$: If $n_r = n_{r-1} = \cdots = n_1 = n_0 = 0$, we have

$$\begin{aligned} &F(a_0 \otimes a_1 \otimes \cdots \otimes a_r) \\ &\quad = \left(d_{r+1} \circ K_r^{\text{old}} + K_{r-1}^{\text{old}} \circ d_r - \text{id} + i_r^{\text{old}} \circ \pi_r^{\text{old}}\right)(a_0 \otimes a_1 \otimes \cdots \otimes a_r) = 0 \end{aligned}$$

by the choice of K^{old}. We prove the general case by induction on the $|n_i|$, in three steps:

FIRST STEP: If $n_r > 0$ and we know that

$$F(a_0 \otimes a_1 \otimes \cdots \otimes a_r t^{n_r - 1}) = 0,$$

or if $n_r < 0$ and we know that

$$F(a_0 \otimes a_1 \otimes \cdots \otimes a_r t^{n_r + 1}) = 0,$$

then

$$F(a_0 \otimes a_1 \otimes \cdots \otimes a_r t^{n_r}) = 0.$$

Proof: We start with the case $n_r > 0$. Since all the maps $d_.$, $K_.^{\text{new}}$, $i_.^{\text{new}}$, $\pi_.^{\text{new}}$ commute with multiplication of the first coordinate in each monomial by t,

$$F(a_0 t \otimes a_1 \otimes \cdots \otimes a_r t^{n_r - 1}) = t \cdot F(a_0 \otimes a_1 \otimes \cdots \otimes a_r t^{n_r - 1}) = 0$$

by the inductive hypothesis.

Therefore, proving that $F(a_0 \otimes a_1 \otimes \cdots \otimes a_r t^{n_r}) = 0$ is equivalent to proving that

$$F(\Delta) = 0$$

for

$$\Delta = a_0 \otimes a_1 \otimes \cdots \otimes a_r t^{n_r} - a_0 t \otimes a_1 \otimes \cdots \otimes a_r t^{n_r - 1}.$$

We have

$$\begin{aligned} K_r^{\text{new}}(\Delta) = (-1)^r &\left[(a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}) * (1 \otimes t)\right] \otimes a_r t^{n_r - 1} \\ &+ a_r t^{n_r - 1} K_{r-1}^{\text{old}}(a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}) * (1 \otimes t), \end{aligned}$$

so

$$(1.15)\quad \begin{aligned} d_{r+1}\circ K_r^{\text{new}}(\Delta) =&(-1)^r\,[b'(a_0\otimes a_1\otimes\cdots\otimes a_{r-1})*(1\otimes t)]\otimes a_r t^{n_r-1}\\ &-a_0t\otimes a_1\otimes\cdots\otimes a_{r-1}\otimes a_r t^{n_r-1}\\ &-[(a_0\otimes a_1\otimes\cdots\otimes a_{r-2})*(1\otimes t)]\otimes a_{r-1}a_r t^{n_r-1}\\ &+a_0\otimes a_1\otimes\cdots\otimes a_{r-1}\otimes a_r t^{n_r}\\ &-(a_r t^{n_r-1}a_0\otimes a_1\otimes\cdots\otimes a_{r-1})*(1\otimes t)\\ &+a_r t^{n_r-1}d_r\circ K_{r-1}^{\text{old}}(a_0\otimes a_1\otimes\cdots\otimes a_{r-1})*(1\otimes t).\end{aligned}$$

Here b' is the operator defined in (1.2); we are using the fact that the Hochschild boundary map $d.$ satisfies the Leibniz rule with respect to the shuffle product $*$, and the fact that $d_1(1\otimes t)=0$.

Now by a similar calculation,

(1.16)

$$\begin{aligned} K_{r-1}^{\text{new}}\circ d_r(\Delta) =&(-1)^{r-1}\,[b'(a_0\otimes a_1\otimes\cdots\otimes a_{r-1})*(1\otimes t)]\otimes a_r t^{n_r-1}\\ &+a_0t\otimes a_1\otimes\cdots\otimes a_{r-1}\otimes a_r t^{n_r-1}\\ &-[(a_0\otimes a_1\otimes\cdots\otimes a_{r-2})*(1\otimes t)]\otimes a_{r-1}a_r t^{n_r-1}\\ &-a_rK_{r-1}^{\text{new}}(0)+a_r t^{n_r-1}K_{r-2}^{\text{old}}(b'(a_0\otimes a_1\otimes\cdots\otimes a_{r-1}))*(1\otimes t)\\ &+(-1)^{r-1}a_r t^{n_r-1}K_{r-2}^{\text{old}}(a_{r-1}a_0\otimes a_1\otimes\cdots\otimes a_{r-2})*(1\otimes t).\end{aligned}$$

So

(1.17)

$$\begin{aligned} (d_{r+1}\circ K_r^{\text{new}}&+K_{r-1}^{\text{new}}\circ d_r)(\Delta)=-a_0t\otimes a_1\otimes\cdots\otimes a_{r-1}\otimes a_r t^{n_r-1}\\ &+a_0\otimes a_1\otimes\cdots\otimes a_{r-1}\otimes a_r t^{n_r}+a_r t^{n_r-1}d_r\circ K_{r-1}^{\text{old}}(a_0\otimes a_1\otimes\cdots\otimes a_{r-1})\\ &+a_r t^{n_r-1}K_{r-2}^{\text{old}}(b'(a_0\otimes a_1\otimes\cdots\otimes a_{r-1}))*(1\otimes t)\\ &+(-1)^{r-1}a_r t^{n_r-1}K_{r-2}^{\text{old}}(a_{r-1}a_0\otimes a_1\otimes\cdots\otimes a_{r-2})*(1\otimes t)\\ =&-a_0t\otimes a_1\otimes\cdots\otimes a_{r-1}\otimes a_r t^{n_r-1}+a_0\otimes a_1\otimes\cdots\otimes a_{r-1}\otimes a_r t^{n_r}\\ &-a_r t^{n_r-1}\left((\mathrm{id}-d_r\circ K_{r-1}^{\text{old}}-K_{r-2}^{\text{old}}\circ d_{r-1})(a_0\otimes a_1\otimes\cdots\otimes a_{r-1})\right)*(1\otimes t)\\ =&\Delta-a_r t^{n_r-1}\left(i_{r-1}^{\text{old}}\circ\pi_{r-1}^{\text{old}}(a_0\otimes a_1\otimes\cdots\otimes a_{r-1})\right)*(1\otimes t)\end{aligned}$$

by the choice of $K^{\text{old}}_{.}$. But since

$$\begin{aligned} i_r^{\text{new}}\circ\pi_r^{\text{new}}(\Delta)&=i_r^{\text{new}}(a_r t^{n_r-1}\pi_{r-1}^{\text{old}}(a_0\otimes a_1\otimes\cdots\otimes a_{r-1})\,\mathrm{dt})\\ &=a_r t^{n_r-1}\left(i_{r-1}^{\text{old}}\circ\pi_{r-1}^{\text{old}}(a_0\otimes a_1\otimes\cdots\otimes a_{r-1})\right)*(1\otimes t),\end{aligned}$$

this calculation shows exactly that

$$F(\Delta) = \left(d_{r+1} \circ K_r^{\text{new}} + K_{r-1}^{\text{new}} \circ d_r - \text{id} + i_r^{\text{new}} \circ \pi_r^{\text{new}}\right)(\Delta) = 0.$$

If $n_r < 0$, we set

$$\Delta = a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1} \otimes a_r t^{n_r+1} - a_0 t \otimes a_1 \otimes \cdots \otimes a_{r-1} \otimes a_r t^{n_r} \tag{1.18}$$

and note that

$$\begin{aligned} K_r^{\text{new}}(\Delta) = (-1)^r \left[(a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}) * (1 \otimes t)\right] \otimes a_r t^{n_r-1} \\ + a_r t^{n_r-1} K_{r-1}^{\text{old}}(a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}) * (1 \otimes t). \end{aligned} \tag{1.19}$$

Similar formulæ hold for $K_r^{\text{new}} \circ d_{(i)}$, $0 \leq i < r$ (and $d_{(r)}(\Delta) = 0$), so the exact argument in formulæ (1.15), (1.16), and (1.17) carries through and gives us $F(\Delta) = 0$. We then use the fact that $F(a_0 \otimes a_1 \otimes \cdots \otimes a_r t^{n_r+1}) = 0$ to obtain

$$F(a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1} \otimes a_r t^{n_r}) = t^{-1} F(a_0 t \otimes a_1 \otimes \cdots \otimes a_{r-1} \otimes a_r t^{n_r}) = 0.$$

SECOND STEP: If $1 \leq j \leq r-1$, and $n_j > 0$ and we know that

$$F(a_0 \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes a_j t^{n_j-1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r}) = 0,$$

or if $n_j < 0$ and we know that

$$F(a_0 \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes a_j t^{n_j+1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r}) = 0,$$

then

$$F(a_0 \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes a_j t^{n_j} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r}) = 0.$$

Proof: As in the first step, we begin by considering the case of $n_j > 0$.

$$\begin{aligned} &F(a_0 t \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes a_j t^{n_j-1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r}) \\ &\quad = t \cdot F(a_0 \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes a_j t^{n_j-1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r}) = 0 \end{aligned}$$

so we set

$$\begin{aligned} \Delta = a_0 \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes a_j t^{n_j-1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r} \\ - a_0 t \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes a_j t^{n_j-1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r} \end{aligned}$$

and prove that $F(\Delta) = 0$.

$$K_r^{\mathrm{new}}(\Delta) = (-1)^j \left[(a_0 \otimes a_1 \otimes \cdots \otimes a_{j-1}) * (1 \otimes t)\right] \otimes a_j t^{n_j - 1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r}$$

and therefore we get that

(1.20)

$$\begin{aligned} & d_{r+1} \circ K_r^{\mathrm{new}}(\Delta) \\ =&(-1)^j [[b'(a_0 \otimes \cdots \otimes a_{j-1}) * (1 \otimes t)] \otimes a_j t^{n_j - 1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r} \\ &+ (-1)^{j-1} a_0 t \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes a_j t^{n_j - 1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r} \\ &+ (-1)^{j+1} \left[(a_0 \otimes \cdots \otimes a_{j-2}) * (1 \otimes t)\right] \otimes a_{j-1} a_j t^{n_j - 1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r} \\ &+ (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes a_j t^{n_j} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r} \\ &+ (-1)^j \left[(a_0 \otimes \cdots \otimes a_{j-1}) * (1 \otimes t)\right] \otimes b'(a_j t^{n_j - 1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r}) \\ &+ (-1)^{r+1} a_r t^{n_r} \left[(a_0 \otimes \cdots \otimes a_{j-1}) * (1 \otimes t)\right] \otimes a_j t^{n_j - 1} \otimes \cdots \otimes a_{r-1} t^{n_{r-1}}] \end{aligned}$$

and that

(1.21)

$$\begin{aligned} & K_{r-1}^{\mathrm{new}} \circ d_r(\Delta) \\ =&(-1)^{j-1} \left[b'(a_0 \otimes \cdots \otimes a_{j-1}) * (1 \otimes t)\right] \otimes a_j t^{n_j - 1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r} \\ &+ \left[(a_0 \otimes \cdots \otimes a_{j-2}) * (1 \otimes t)\right] \otimes a_{j-1} a_j t^{n_j - 1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r} \\ &- \left[(a_0 \otimes \cdots \otimes a_{j-1}) * (1 \otimes t)\right] \otimes b'(a_j t^{n_j - 1} \otimes a_{j+1} t^{n_{j+1}} \otimes \cdots \otimes a_r t^{n_r}) \\ &+ (-1)^{j+r} a_r t^{n_r} \left[(a_0 \otimes \cdots \otimes a_{j-1}) * (1 \otimes t)\right] \otimes a_j t^{n_j - 1} \otimes \cdots \otimes a_{r-1} t^{n_{r-1}}. \end{aligned}$$

Adding the last two equations, we get

$$(d_{r+1} \circ K_r^{\mathrm{new}} + K_{r-1}^{\mathrm{new}} \circ d_r)(\Delta) = \Delta.$$

Inspection of the formulæ for i_r^{new} and π_r^{new} show that

$$i_r^{\mathrm{new}} \circ \pi_r^{\mathrm{new}}(\Delta) = 0$$

and thus we obtain the desired equality $F(\Delta) = 0$.

When $n_r < 0$, we change the definition of Δ as we did in (1.18); the calculation analogous to (1.19) holds, and so the proof described in (1.20) and (1.21) gives $F(\Delta) = 0$.

THIRD STEP: For any $n_0 \geq 0$,

$$F(a_0 t^{n_0} \otimes a_1 t^{n_1} \otimes \cdots \otimes a_r t^{n_r}) = 0.$$

Proof:

$$F(a_0 t^{n_0} \otimes a_1 t^{n_1} \otimes \cdots \otimes a_r t^{n_r}) = t^{n_0} F(a_0 \otimes a_1 t^{n_1} \otimes \cdots \otimes a_r t^{n_r}) = 0$$

by the previous steps. ∎

COROLLARY 1: *If the complex $X_.$ is a deformation retract of the reduced bar complex calculating $HH_*(A)$ for A an algebra as described before* (1.1), *then the complex*

$$\cdots \xrightarrow{d} X_3[t] \oplus X_2[t]\,\mathrm{dt} \xrightarrow{d} X_2[t] \oplus X_1[t]\,\mathrm{dt} \xrightarrow{d} X_1[t] \oplus X_0[t]\,\mathrm{dt} \xrightarrow{d} X_0[t] \longrightarrow 0$$

with d induced by d_X as in (1.8) is a deformation retract of the reduced bar complex calculating $HH_*(A[t])$.

COROLLARY 2: *If the complex $X_.$ is a deformation retract of the reduced bar complex calculating $HH_*(A)$ for A an algebra as described before (1.1), then the complex*

$$\cdots \xrightarrow{d} X_2[t,t^{-1}] \oplus X_1[t,t^{-1}]\,\mathrm{dt} \xrightarrow{d} X_1[t,t^{-1}] \oplus X_0[t,t^{-1}]\,\mathrm{dt} \xrightarrow{d} X_0[t,t^{-1}] \longrightarrow 0$$

with d induced by d_X as in (1.8) is a deformation retract of the reduced bar complex calculating $HH_*(A[t,t^{-1}])$.

2. Going from a ring A to $C^\infty[S^1;A]$

In this section we start with a Banach algebra A, and consider the extension of A which we get by taking $C^\infty[S^1;A]$, the space of infinitely differentiable maps from the circle to A. When equipped with the metric constructed from the Sobolev-type norms (supremums of f and finite numbers of derivative), and with pointwise multiplication, $C^\infty[S^1;A]$ becomes a Fréchet algebra. Note that in this case the construction cannot, therefore, be iterated. A is embedded in $C^\infty[S^1;A]$ as the sub-algebra of all constant functions.

The Hochschild homology of such an algebra A is huge, and most of the classes in it are not boundaries because the 'elements' which 'should' map to them by the boundary map of the usual Hochshcild complex are actually infinite sums of tensored monomials. To correct this, we will use continuous Hochschild homology, which is defined with the topological projective tensor product $\hat{\otimes}_\pi$ in place of the usual $\otimes$ (see [2], section 5.6.2). $\hat{\otimes}_\pi$ has the property that

$$C^\infty(V)\hat{\otimes}_\pi C^\infty(V) \cong C^\infty(V \times V) \tag{2.1}$$

for any compact smooth manifold V (where $C^\infty(V)$ refers to the ring of smooth complex-valued functions). Fourier expansion shows that for any n-torus $\mathbb{T}^n$ and any Banach algebra A,

$$C^\infty(\mathbb{T}^n; A) \cong C^\infty(\mathbb{T}^n)\hat{\otimes}_\pi A. \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$C^\infty[S^1; A]^{\hat{\otimes}_\pi (r+1)} \cong C^\infty(\mathbb{T}^{r+1}; \underbrace{A\hat{\otimes}_\pi A\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi A}_{r+1 \text{ times}}).$$

We define

$$\begin{aligned} s_i\colon C^\infty[S^1; A]^{\hat{\otimes}_\pi (r)} &\cong C^\infty(\mathbb{T}^r; \underbrace{A\hat{\otimes}_\pi A\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi A}_{r \text{ times}}) \\ &\longrightarrow C^\infty[S^1; A]^{\hat{\otimes}_\pi (r+1)} \cong C^\infty(\mathbb{T}^{r+1}; \underbrace{A\hat{\otimes}_\pi A\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi A}_{r+1 \text{ times}}), \end{aligned}$$

for $0 \le 1 \le r-1$, by letting

$$\begin{aligned} &(s_i(f_0 \otimes f_1 \otimes \cdots \otimes f_{r-1}))(t_0, t_1, \ldots, t_r) \\ &\qquad = f_0(t_0) \otimes f_1(t_1) \otimes \cdots \otimes f_i(t_i) \otimes 1 \otimes f_{i+1}(t_{i+2}) \otimes \cdots \otimes f_{r-1}(t_r). \end{aligned}$$

This gives us the formula

$$C^\infty[S^1; A]\hat{\otimes}_\pi \overline{C^\infty[S^1; A]}^{\hat{\otimes}_\pi (r)} \cong C^\infty(\mathbb{T}^{r+1}; A\hat{\otimes}_\pi A\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi A) / \left(\bigcup_{i=0}^{r-1} \operatorname{Im}(s_i) \right)$$

for the modules appearing in the reduced bar complex.

The boundary map for this reduced bar complex is $d_r = \sum_{i=0}^r (-1)^i d_{(i)}$, where for $0 \le i < r$,

$$\begin{aligned} &(d_{(i)}(f_0 \otimes \cdots \otimes f_r))(t_0, t_1, \ldots, t_{r-1}) \\ &\qquad = f_0(t_0) \otimes f_1(t_1) \otimes \cdots \otimes f_i(t_i) f_{i+1}(t_i) \otimes f_{i+2}(t_{i+1}) \otimes \cdots \otimes f_r(t_{r-1}) \end{aligned}$$

and

$$(d_{(r)}(f_0\otimes\cdots\otimes f_r))(t_0,t_1,\ldots,t_{r-1})=f_r(t_0)f_0(t_0)\otimes f_1(t_1)\otimes\cdots\otimes\cdots\otimes f_{r-1}(t_{r-1}).$$

The argument which shows that the reduced bar complex has the same homology as the standard bar complex when we use $\hat{\otimes}_\pi$ is identical to the argument when we use $\otimes$— the kernels of the reduction map form an acyclic complex.

We assume, as usual, that we have a complex $X_.$ which is a direct summand in the reduced bar complex of A, quasi-isomorphic to the whole but easier to calculate with, along with inclusion and projection maps $i_.^{\mathrm{old}}$ and $\pi_.^{\mathrm{old}}$ and a homotopy $K_.^{\mathrm{old}}$.

We construct a new complex and prove its homology to be $\mathrm{HH}_*(C^\infty[S^1;A])$, the complex

$$(2.3)\qquad \begin{aligned}\cdots\xrightarrow{d}C^\infty(S^1;X_2)\oplus C^\infty(S^1;X_1)\,\mathrm{dt}\xrightarrow{d}C^\infty(S^1;X_1)\\ \oplus\, C^\infty(S^1;X_0)\,\mathrm{dt}\xrightarrow{d}C^\infty(S^1;X_0)\longrightarrow 0\end{aligned}$$

where for all $f\colon S^1\longrightarrow X_.$, $t\in S^1$

$$(d(f))(t)=d_X(f(t)),\qquad (d(f\,\mathrm{dt}))(t)=d_X(f(t))\,\mathrm{dt}.$$

dt is again a formal place marker.

$i_.^{\mathrm{new}}$ and $\pi_.^{\mathrm{new}}$ are also defined analogously to what we have done before— for $f\colon S^1\longrightarrow X_r$ and $(t_0,t_1,\ldots,t_r)\in\mathbb{T}^{r+1}$,

$$(i_r^{\mathrm{new}}(f))(t_0,t_1,\ldots,t_r)=i_r^{\mathrm{old}}(f(t_0))$$

and if we write $i_r^{\mathrm{old}}(f(t_0))=\sum_{a=1}^{\ell}b_0^{(a)}\otimes b_1^{(a)}\otimes\cdots\otimes b_r^{(a)}$,

$$\begin{aligned}&(i_r^{\mathrm{new}}(f\,\mathrm{dt}))(t_0,t_1,\ldots,t_r,t_{r+1})\\ &\quad=\sum_{i=0}^{r}(-1)^{r-i}\sum_{a=1}^{\ell}b_0^{(a)}\otimes b_1^{(a)}\otimes\cdots\otimes b_i^{(a)}\otimes t_{i+1}\otimes b_{i+1}^{(a)}\otimes\cdots\otimes b_r^{(a)}\end{aligned}$$

where each t_{i+1} is regarded as an element of A via the inclusion of S^1 in $\mathbb{C}$ as the unit circle and the inclusion of $\mathbb{C}$ in A as multiples of the unit element of A.

For a monomial $f_0\otimes f_1\otimes\cdots\otimes f_r\colon\mathbb{T}^{r+1}\longrightarrow A\hat{\otimes}_\pi A\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi A$, we let

$$\begin{aligned}&(\pi_r^{\mathrm{new}}(f_0\otimes f_1\otimes\cdots\otimes f_r))(t)\\ &=\pi_r^{\mathrm{old}}(f_0(t)\otimes f_1(t)\otimes\cdots\otimes f_r(t))\\ &\quad+\frac{\mathrm{d}f_r}{\mathrm{dt}}(t)\pi_{r-1}^{\mathrm{old}}(f_0(t)\otimes f_1(t)\otimes\cdots\otimes f_{r-1}(t))\,\mathrm{dt}.\end{aligned}$$

The desired relation,

$$\pi_r^{\mathrm{new}} \circ i_r^{\mathrm{new}} = \mathrm{id}_{C^\infty(S^1;X_r)\oplus C^\infty(S^1;X_{r-1})\,\mathrm{dt}},$$

still holds, by calculations exactly analogous to (1.11) and (1.12), but it is a bit harder to see this. The point is that in our definition of π_r^{new} above, we are looking at monomials that take $(t_0, t_1, \ldots, t_r)$ to $f_0(t_0) \otimes f_1(t_1) \otimes \cdots \otimes f_r(t_r)$, and the form in which $i_r^{\mathrm{new}}(f)$ is given is misleading in that it does not consist of monomials of that form. To get it into that form, we would take $i_r^{\mathrm{new}}(f(t_0))$ and break it, using Fourier expansion, into monomials of the form

$$g_0(t_0) \otimes a_1 t_0^{n_1} \otimes a_2 t_0^{n_2} \otimes \cdots \otimes a_r t_0^{n_r} = g_0(t_0) t_0^{n_1+n_2+\cdots n_r} \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_r$$

(using the fact that t_0 is just a complex number which can pass through the tensor). This presentation makes it clear why, on the image of i_r^{new}, $\frac{\mathrm{d}f_r}{\mathrm{dt}} = 0$ unless $f_r = t_r$.

It remains to show that $i_r^{\mathrm{new}} \circ \pi_r^{\mathrm{new}} \simeq \mathrm{id}$, by finding a suitable homotopy $K_\cdot^{\mathrm{new}}$. As motivation for the construction, note that we can view $A[t, t^{-1}]$ as embedded into $C^\infty[S^1; A]$, by looking at t as the inclusion map $S^1 \hookrightarrow \mathbb{C} \cdot 1_A$ and at t^{-1} as the composition $S^1 \xrightarrow{t \mapsto 1/t} S^1 \hookrightarrow \mathbb{C} \cdot 1_A$. Moreover, because $A[t, t^{-1}]$ consists of finite polynomials, its (regular) tensor products with itself embed in the topological projective tensor product of $C^\infty[S^1; A]$ with itself, and all the constructions and maps we have defined commute with these embeddings. Thus to find the homotopy $K_\cdot^{\mathrm{new}}$ for $C^\infty[S^1; A]$ we adapt the homotopy which was used in the previous section.

For any C^∞ function f on the circle, we can use the Fourier expansion $f(t) = \sum_{n=-\infty}^{\infty} a_n t^n$ to write

$$f = f^- + a_0 + f^+$$

where

$$f^-(t) = \sum_{n=-\infty}^{-1} a_n t^n, \quad f^+(t) = \sum_{n=1}^{\infty} a_n t^n.$$

This is necessary because of the asymmetry between t and t^{-1} in the formulæ of the previous section. We set

$$\begin{aligned}
&(K_r^{\mathrm{new}}(f_0\otimes f_1\otimes\cdots\otimes f_r))(t_0,t_1,\dots,t_r)\\
&=\sum_{j=1}^{r}\sum_{h=0}^{j-1}(-1)^{h+1}\Big[f_0(t_0)\otimes f_1(t_0)\otimes\cdots\otimes f_h(t_0)\otimes t_{h+1}\otimes f_{h+1}(t_0)\\
&\qquad\otimes\cdots\otimes f_{j-1}(t_0)\otimes\frac{f_j^+(t_0)-f_j^+(t_{j+1})}{t_0-t_{j+1}}\otimes f_{j+1}(t_{j+2})\otimes\cdots\otimes f_r(t_{r+1})\\
&\quad-f_0(t_0)\otimes f_1(t_0)\otimes\cdots\otimes f_h(t_0)\otimes t_{h+1}\otimes f_{h+1}(t_0)\\
&\qquad\otimes\cdots\otimes f_{j-1}(t_0)\otimes\frac{f_j^-(t_0)-f_j^-(t_{j+1})}{t_0-t_{j+1}}\otimes f_{j+1}(t_{j+2})\otimes\cdots\otimes f_r(t_{r+1})\Big]\\
&\quad+\frac{\mathrm{d}f_r}{\mathrm{dt}}(t_0)\sum_{h=0}^{r-1}(-1)^{r-1-h}\sum_{a=0}^{\ell}b_0^{(a)}\otimes\cdots\otimes b_h^{(a)}\otimes t_{h+1}\otimes b_{h+1}^{(a)}\otimes\cdots\otimes b_r^{(a)}\\
&\quad+K_r^{\mathrm{old}}(f_0(t_0)\otimes f_1(t_0)\otimes\cdots\otimes f_r(t_0))
\end{aligned}$$

where the difference quotients $(f(t_0)-f(t_{j+1}))/(t_0-t_{j+1})$ are defined as $f'(t_0)$ if $t_0=t_{j+1}$, and

$$K_{r-1}^{\mathrm{old}}(f_0(t_0)\otimes f_1(t_0)\otimes\cdots\otimes f_{r-1}(t_0))=\sum_{a=0}^{\ell}b_0^{(a)}\otimes\cdots\otimes b_1^{(a)}\otimes\cdots\otimes b_r^{(a)}.$$

This definition coincides with (1.4) on monomials $f_0\otimes f_1\otimes\cdots\otimes f_r$ where $f_i(t)=a_it^{n_i}$ for all $0\le i\le r$ and $t\in S^1$, so we know that on the copy of $A[t,t^{-1}]\otimes\overline{A[t,t^{-1}]}^{\otimes(r)}$ which is embedded inside $C^\infty[S^1;A]\otimes\overline{C^\infty[S^1;A]}^{\hat{\otimes}_\pi(r)}$ we have

$$d_{r+1}\circ K_r^{\mathrm{new}}+K_{r-1}^{\mathrm{new}}\circ d_r=\mathrm{id}-i_r^{\mathrm{new}}\circ\pi_r^{\mathrm{new}}. \tag{2.4}$$

Note however that all the functions in this equation are continuous as functions from the reduced bar complex to itself (with respect to the Fréchet topology). Fourier expansion shows us that the embedded copy of $A[t,t^{-1}]\otimes\overline{A[t,t^{-1}]}^{\otimes(r)}$ on which (2.4) holds is dense in $C^\infty[S^1;A]\otimes\overline{C^\infty[S^1;A]}^{\hat{\otimes}_\pi(r)}$. We deduce that (2.4) holds on all of $C^\infty[S^1;A]\otimes\overline{C^\infty[S^1;A]}^{\hat{\otimes}_\pi(r)}$, and so $K^{\mathrm{new}}_{\cdot}$ is indeed the desired homotopy.

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